A story of three hedgehogs

Here they are:

- The quotient hedgehog: Let the closed unit interval $I = [0,1]$ be equipped with the standard topology, $\mathbb{N}$ with the discrete topology, and $P = I \times \mathbb{N}$ with the Tychonoff product topology. Consider the equivalence relation $\sim$ on $P$ whose only non-trivial equivalence class is $\{0\} \times \mathbb{N}$ (so any point $(a,n) \in P$ with $a \neq 0$ is equivalent only to itself). The quotient hedgehog is the set $X = P/\sim$ equipped with the quotient topology.

- The metric hedgehog: Let $J = (0,1]$ be the semi-open unit interval. Put $Y = \{0\} \cup (J \times \mathbb{N})$. Consider the metric $d$ on $Y$ given by
  
  - $d((x,n),(y,m)) = |x - y|$;
  - $d((x,n),(y,m)) = x + y$ if $n \neq m$;
  - $d((x,n),0) = x$.

  The metric hedgehog is $Y$ equipped with the topology generated by $d$.

- The compact hedgehog: Consider $I$ with the standard topology and $I^\mathbb{N}$ with the Tychonoff product topology. Put $Z = \{x \in I^\mathbb{N} : x(k) = 0 \text{ for all but at most one } k \in \mathbb{N}\}$. The compact hedgehog is the set $Z$ equipped with the topology inherited from $I^\mathbb{N}$.

Problem 1. Fill the table on page 2 with + and - for “yes” or “no”. Use extra paper to write supporting argument in difficult cases.

Problem 2. Recall that a condensation of one topological space onto another one is a continuous bijection (and in this case, we say that the first space condenses onto the second). The three hedgehogs say:

- A: I am the best! My topological space condenses onto each of the two others!
- B: I am strong, too. My topological space condenses onto one of the two others (but not to the other one).
- C: I am sorry, my topological space does not condense onto any of the two others.

Assuming that all what they say is true, determine who are A, B, and C. Use extra paper to write supporting argument.

Problem 3. If you like, write whatever else you think might be appropriate about these three spaces.

SOLUTION
<table>
<thead>
<tr>
<th>Property</th>
<th>quotient hegehog</th>
<th>metric hegehog</th>
<th>compact hegehog</th>
</tr>
</thead>
<tbody>
<tr>
<td>second countable</td>
<td>-</td>
<td>+</td>
<td>+</td>
</tr>
<tr>
<td>first countable</td>
<td>-</td>
<td>+</td>
<td>+</td>
</tr>
<tr>
<td>Fréchet</td>
<td>+</td>
<td>+</td>
<td>+</td>
</tr>
<tr>
<td>metrizable</td>
<td>-</td>
<td>+</td>
<td>+</td>
</tr>
<tr>
<td>Lindelöf</td>
<td>+</td>
<td>+</td>
<td>+</td>
</tr>
<tr>
<td>compact</td>
<td>-</td>
<td>-</td>
<td>+</td>
</tr>
<tr>
<td>countably compact</td>
<td>-</td>
<td>-</td>
<td>+</td>
</tr>
<tr>
<td>separable</td>
<td>+</td>
<td>+</td>
<td>+</td>
</tr>
<tr>
<td>Hausdorff</td>
<td>+</td>
<td>+</td>
<td>+</td>
</tr>
<tr>
<td>A, B, or C?</td>
<td>A</td>
<td>B</td>
<td>C</td>
</tr>
</tbody>
</table>
The quotient hedgehog $X$ is not first countable. (The proof repeats the proof for the Arens space.) Therefore $X$ is not second countable and not metrizable.

$X$ is Fréchet. Let $s \in X$ be the point "where the needles meet" (the class of equivalence of $<0, n>$). We have to prove that for every $A \subset X$ and $a \in X$, if $a \neq \overline{A}$, then there is a sequence converging from $A$ to $a$.

Every point a different from $s$ has a neighborhood homeomorphic to the open interval of $\mathbb{R}$, and we know that $\mathbb{R}$ is Fréchet. So let $a = s$. Let $q : P \to X$ be the quotient mapping.

Claim: If $s \in A \setminus A$ then $\exists n \in \mathbb{N}$ s.t. $s \in q^{-1}(\mathbb{R} \times \{n\})$

Indeed, otherwise for every $n \in \mathbb{N}$, there is $\varepsilon_n > 0$ such that $q \left( (0, \varepsilon_n) \times \{n\} \right) \cap A = \emptyset$. But then $\{s \in q^{-1}(U) \mid U \text{ neighborhood of } s \text{ in } X \}$ does not intersect $A$. A contradiction.

Let $n$ be as in Claim. It remains to note that $s \in q^{-1}(\mathbb{R} \times \{n\})$ and $q^{-1}(\mathbb{R} \times \{n\})$ is homeomorphic to $I$, and we know that $I$ is Fréchet.

Let $P = I \times \mathbb{N}$ be the union of countably many separable Lindelöf subspaces. Hence $P$ is separable and separable Lindelöf. Hence so is $X$ being a continuous image of $P$.

$X$ is not countably compact: $q(\mathbb{R} \times \mathbb{N})$ is an infinite closed discrete subspace of $X$. Hence $X$ is not compact.

$X$ is Hausdorff - easy to see.
The metric hedgehog $Y$ is metrizable by definition (it is easy to see that $d$ satisfies all 3 axioms of metric).

Each $X \times \mathbb{R}$ is separable being homeomorphic to a subspace of the real line. Therefore $Y$ is separable. Being metrizable, it follows that it is Lindelöf, first countable and second countable.

$Y$ is not countably compact (similar to $X$) and thus not compact. As any metrizable space, $Y$ is Hausdorff.

The compact hedgehog $Z$. $Z$ is a closed subspace of the compact space $I^{\mathbb{N}}$ (see section of the notes on compactness), so $Z$ is compact hence countably compact and Lindelöf.

$Z$ is a subspace of a metrizable space $I^{\mathbb{N}}$, so $Z$ is metrizable, hence has all properties in the list (for metrizable spaces, Lindelöf'sness is equivalent to separability and second countability).

$A = X, \quad B = Y, \quad C = Z$.

A condensation $p: X \to Y$ is defined as $p(s) = 0$ and $p(\langle x, n \rangle) = \langle x, n \rangle$ for $x \in J, \ n \in \mathbb{N}$.

A condensation $r: Y \to Z$ is defined by

$\Gamma(0) = 0$, a point with all coordinates 0

$\Gamma(\langle x, n \rangle) = \langle 0, \ldots, 0, x, 0, \ldots \rangle$ where $x \in J, \ n \in \mathbb{N}$.