Mappings of topological spaces

1. DEFINITIONS

Recall that a mapping \( f : X \to Y \) is

- a **surjection** (or an onto mapping) if for every \( y \in Y \) there is at least one \( x \in X \) such that \( f(x) = y \);
- an **injection** (or a 1-1 mapping) if \( x \neq y \implies f(x) \neq f(y) \);
- a **bijection** (or a 1-1 correspondence) if \( f \) is both a surjection and an injection.

Recall also that \( f \) is a bijection iff there is the inverse function \( f^{-1} : Y \to X \).

**Definition 1.** Let \((X, T_X)\) and \((Y, T_Y)\) be topological spaces and \( f : X \to Y \) a mapping. \( f \) is called

- **continuous** if \( V \in T_Y \implies f^{-1}(V) \in T_X \);
- **open** if \( U \in T_X \implies f(U) \in T_Y \);
- **closed** if \( F \) is closed in \( X \implies f(F) \) is closed in \( Y \);
- **quotient** if \( V \in T_Y \iff f^{-1}(V) \in T_X \);
- **condensation** if \( f \) is a continuous bijection;
- **homeomorphism** if \( f \) is a quotient mapping and a bijection;
- a **homeomorphic imbedding into** \((Y, T_Y)\) if the restriction \( f : X \to f(X) \) is a homeomorphism.

In the diagram below, arrows mean implications.

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\[
\text{homeomorphism} \quad \text{continuous and closed} \quad \text{continuous and open} \quad \text{condensation} \quad \text{quotient} \quad \text{continuous}
\]

2. CONTINUOUS Mappings

**Exercise 2.** (1) Show that any mapping from a discrete space to any space is continuous.\(^2\)

(2) Show that any mapping from any space to a space with the trivial topology is continuous.\(^3\)

(3) Show that a **constant function** from any space \((X, T_X)\) to any other space \((Y, T_Y)\), that is such a function that for some \( c \in Y \), \( f(x) = c \) for all \( x \in X \) is continuous.

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\(^1\)Usually, this property is considered only for surjective mappings \( f \)

\(^2\)Moreover, the discrete topology is the only topology with this property.

\(^3\)Moreover, the trivial topology is the only topology with this property.
Proposition 3. Let \( f : X \rightarrow Y \) where \((X, T_X)\) and \((Y, T_Y)\) are topological spaces. The following conditions are equivalent:

1. \( f \) is continuous;
2. for every \( V \in \mathcal{B} \), where \( \mathcal{B} \) is some base of \( Y \), \( f^{-1}(V) \in T_X \);
3. for every \( x_0 \in X \) for every neighborhood \( V \in T_Y \) with \( V \ni f(x_0) \), there is \( U \in T_X \) such that \( U \ni x_0 \) and \( f(U) \subset V \);
4. \( H \subset Y \) is closed \( \Rightarrow f^{-1}(H) \) is closed;
5. for every \( A \subset X \), \( f(A) \subset f(A) \).

Moreover,

Proposition 4. If \((X, T_X)\) and \((Y, T_Y)\) are Fréchet spaces, then the following conditions are equivalent:

1. \( f : X \rightarrow Y \) is continuous;
2. If a sequence \( \{a_n : n \in \mathbb{N}\} \) of points of \( X \) converges to \( x \in X \), then the sequence \( \{f(a_n) : n \in \mathbb{N}\} \) converges to \( f(x) \).

For mappings from \( \mathbb{R} \) to \( \mathbb{R} \), the topological definition of continuity matches the well known definition from analysis:

Proposition 5. A mapping \( f : \mathbb{R} \rightarrow \mathbb{R} \) (where \( \mathbb{R} \) bears the standard topology) is continuous iff \( (\forall x_0 \in \mathbb{X}) \ (\forall \varepsilon > 0) \ (\exists \delta > 0) \) such that \( (\forall x \in \mathbb{X}) \ |x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon \).

3. Open mappings and closed mappings

Exercise 6. Show that any mapping from any space to a discrete space is both open and closed. Is such a mapping necessarily continuous?

Proposition 7. If \( X \times Y \) is the Tychonoff product of \((X, T_X)\) and \((Y, T_Y)\) then the projections \( \pi_1 : X \times Y \rightarrow X \) and \( \pi_2 : X \times Y \rightarrow Y \) are continuous and open mappings.

Example 8. A mapping which is continuous and open, but is not closed.

Consider \( \pi_1 : \mathbb{R}^2 \rightarrow \mathbb{R} \). This is a continuous and open mapping. Let \( F \) be the closed region above a curve with two vertical asymptotes. For example, let \( g \) be given by \( g(x) = \frac{1}{x(x-1)} \) and \( F = \{(x,y) \in \mathbb{R}^2 : 0 < x < 1 \text{ and } y \geq g(x) \} \). Then \( F \) is a closed subspace of \( \mathbb{R}^2 \) but the set \( \pi_1(F) = (0,1) \) is not closed in \( \mathbb{R} \).

Example 9. A mapping which is continuous and closed, but not open.

Define \( f : \mathbb{R} \rightarrow \mathbb{R} \) by

\[
 f(x) = \begin{cases} 
 -x & \text{if } x < 0 \\
 0 & \text{if } 0 \leq x \leq 3 \\
 x - 3 & \text{if } x \geq 3 
\end{cases}
\]

Then \( f \) is continuous and closed but not open because \( f((1,2)) = \{0\} \) is not open in \( \mathbb{R} \).

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4If this condition holds at one specific point \( x_0 \) then we say that \( f \) is continuous at \( x_0 \). In this sense, continuous \( \Rightarrow \) continuous at every point.

5Actually, as we will see later, also in the more general case of metric spaces.

6We may discuss details in class.
4. Homeomorphisms and homeomorphic imbeddings

Proposition 10. Let \( f : X \to Y \) be a bijection where \((X, T_X)\) and \((Y, Y)\) are topological spaces. Then the following conditions are equivalent:

1. \( f \) is an homeomorphism;
2. for every \( H \subset Y \), \( H \) closed in \( Y \), iff \( f^{-1}(H) \) is closed in \( X \);
3. for every \( U \subset X \), \( U \in T_X \) iff \( f(U) \in T_Y \);
4. for every \( F \subset X \), \( F \) is closed in \( X \) iff \( f(F) \) is closed in \( Y \);
5. both \( f \) and \( f^{-1} \) are continuous.

If there is a homeomorphism \( f : X \to Y \) then we say that \( X \) and \( Y \) are homeomorphic (or equivalent in the topological sense). Indeed, being homeomorphic is an equivalence relation.

There are two trivial (but important) examples of homeomorphic imbeddings.

Examples 11. (1) If \((X, T)\) is a topological space, and \(Y \subset X\) bears the subspace topology inherited from \(X\), then \(i : Y \to X\) given by \(i(y) = y\) is a homeomorphic imbedding.
(2) If \((X, T_X)\) and \((Y, T_Y)\) are topological spaces, \(X \times Y\) the Tychonoff product, and \(c \in Y\), then \(x \mapsto \langle x, c \rangle\) defines a topological embedding of \(X\) into \(X \times Y\); the image of \(X\) is \(X \times \{c\}\).

5. Condensations; discrete sums of spaces

If there is a condensation from \((X, T_X)\) onto \((Y, T_Y)\), then we say that \((X, T_X)\) condenses onto \((Y, T_Y)\).

Exercise 12. Show that a simi-open interval \([0, 1)\) condenses onto the unit circle \(S^1\).

If \(T_1\) and \(T_2\) are two topologies on the same set \(X\), and \(T_1 \supset T_2\), then the identity mapping \(id : X \to X\) given by \(id(x) = x\) is a condensation of \((X, T_1)\) onto \((X, T_2)\). However, not necessarily a condensation is given by the identity mapping.

Example 13. There are topological spaces \((X, T_X)\) and \((Y, T_Y)\) such that:

- there is a condensation from \(X\) onto \(Y\);
- there is a condensation from \(Y\) onto \(X\);
- \(X\) and \(Y\) are not homeomorphic.

To construct \(X\) and \(Y\) as in Example 13, we need one more construction.

Definition 14. Let \(\mathcal{A}\) be an index set and let for each \(a \in \mathcal{A}\), \((X_a, T_a)\) be a topological space. We assume that the sets \(X_a\) are pairwise disjoint. The discrete sum \((X, T) = \bigoplus_{a \in \mathcal{A}} (X_a, T_a)\) is defined as follows: \(X = \bigcup_{a \in \mathcal{A}} X_a\); a set \(U \subset X\) belongs to \(T\) iff \(U \cap X_a \in T_a\) for each \(a \in \mathcal{A}\).

Exercise 15. (1) Show that the discrete sum of the single point space and a copy of the unit open interval \((0, 1)\) condenses onto a simi-open interval \((0, 1]\).
(2) Show that the discrete sum of the single point space and a copy of the unit open interval \((0, 1)\) condenses onto the unit circle \(S^1\).

Exercise 16. Let \(X\) be the discrete sum of
- countably many isolated points;
- countably many homeomorphic copies of the open interval \((0, 1)\).
• countably many homeomorphic copies of the unit circle $S^1$.

Let $Y$ be the discrete sum of
• countably many isolated points;
• countably many homeomorphic copies of the open interval $(0, 1)$;
• countably many homeomorphic copies of the semi-open interval $[0, 1)$;
• countably many homeomorphic copies of the unit circle $S^1$.

Show that these $X$ and $Y$ are good for Example 13.

6. Quotient sets, quotient spaces, and quotient mappings

Recall that if $\sim$ is an equivalence relation\(^7\) on a set $X$, then for $x \in X$ the equivalence class of $x$ with respect to $\sim$ is the set $\hat{x} = \{x' \in X : x \sim x'\}$. The set of all classes of equivalence $\{\hat{x} : x \in X\}$ is a partition of $X$; it is denoted $X/\sim$ and is called the quotient set of $X$ with respect to $\sim$. The quotient mapping $q : X \rightarrow X/\sim$ is defined by $x \mapsto \hat{x}$.

Now, if $(X, T)$ is a topological space, and $\sim$ is an equivalence relation on the set $X$ then the quotient topology $T_\sim$ on the quotient set $X/\sim$ is defined as follows: for $V \subset X/\sim$, $V \in T_\sim \iff f^{-1}(V) \in T_X$. Note that this is equivalent to $q$ being quotient in the sense of Definition 1.

Conversely, given a mapping $f : X \rightarrow Y$, one can consider the partition of $X$ into the fibers of $f$: $\{f^{-1}(y) : y \in Y\}$ which can be denoted $X/\sim_f$. If $(X, T_X)$ and $(Y, T_Y)$ are topological spaces, then the mapping $f$ is quotient iff the correspondence $y \leftrightarrow f^{-1}(y)$ between $(Y, T_Y)$ and $(X/\sim_f, T_{\sim_f})$ is a homeomorphism.

**Proposition 17.** Let $(X, T_X)$ and $(Y, T_Y)$ be topological spaces and $f : X \rightarrow Y$ a surjection. The following conditions are equivalent:

1. $f$ is a quotient mapping;
2. $T_Y$ is the strongest topology on $Y$ that makes $f$ continuous.

**Proposition 18.** Let $(X, T_X)$ and $(Y, T_Y)$ be topological spaces and $f : X \rightarrow Y$ a bijection. The following conditions are equivalent:

1. $f$ is a quotient mapping;
2. $f$ is an homeomorphism.

**Examples 19.**

1. Consider $X = [0, 2\pi]$ with the standard topology, and let $\sim$ be the equivalence relation on $X$ such that $0 \sim 2\pi$ and the equivalence class of any point $x$ other than $0$ or $2\pi$ is just $\{x\}$. Then $X/\sim$ (with the quotient topology) is homeomorphic to the unit circle $S^1$.

2. Consider $\mathbb{R}$ with the standard topology, and let $\sim$ be the equivalence relation on $\mathbb{R}$ defined by $x \sim x'$ iff $x - x' = 2\pi k$ for some $k \in \mathbb{Z}$. Then $X/\sim$ is homeomorphic to the unit circle $S^1$.\(^7\)

3. Consider the square $X = [0, 2\pi] \times [0, 2\pi]$ and let $\sim$ be the equivalence relation on $X$ such that for every $y \in [0, 2\pi]$, $(0, y) \sim (2\pi, y)$ and the equivalence class of any other point $(x, y)$ is just $\{(x, y)\}$. Then $X/\sim$ is homeomorphic to the cylinder $S^1 \times [0, 2\pi]$.

4. Consider the square $X = [0, 2\pi] \times [0, 2\pi]$ and let $\sim$ be the equivalence relation on $X$ such that:

- for every $y \in [0, 2\pi]$, $(0, y) \sim (2\pi, y)$;

\(^7\)That is, a symmetric, reflexive and transitive binary relation
• for every $x \in [0, 2\pi]$, $\langle x, 0 \rangle \sim \langle x, 2\pi \rangle$;
• the equivalence class of any other point $\langle x, y \rangle$ is just $\{\langle x, y \rangle\}$. Then $X/\sim$ is homeomorphic to the torus $S^1 \times S^1$.

(5) Consider the square $X = [0, 2\pi] \times [0, 2\pi]$ and let $\sim$ be the equivalence relation on $X$ such that:
• for every $y \in [0, 2\pi]$, $(0, y) \sim (2\pi, y)$;
• for every $x \in [0, 2\pi]$, $(x, 0) \sim (2\pi - x, 2\pi)$;
• the equivalence class of any other point $\langle x, y \rangle$ is just $\{\langle x, y \rangle\}$. Then $X/\sim$ is homeomorphic to the space called the Klein bottle.

(6) Consider the closed unit disc $D = \{(x, y) : x^2 + y^2 \leq 1\}$ with the subspace topology inherited from $\mathbb{R}^2$. Let $\sim$ be the equivalence relation on $D$ such that each point $\langle x, y \rangle$ in the boundary circle $S^1$ is equivalent to the symmetric point $(-x, -y)$ and the equivalence class of any point $\langle x, y \rangle$ in the interior of $D$ is just $\{\langle x, y \rangle\}$. Then $X/\sim$ is homeomorphic to the space called the projective plain.

(7) Consider the closed unit disc $D = \{(x, y) : x^2 + y^2 \leq 1\}$ with the subspace topology inherited from $\mathbb{R}^2$. Let $\sim$ be the equivalence relation on $D$ such that the boundary circle $S^1$ forms one class of equivalence and the equivalence class of any point $\langle x, y \rangle$ in the interior of $D$ is just $\{\langle x, y \rangle\}$. Then $X/\sim$ is homeomorphic to the sphere $S^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$.

(8) Consider the set $X = \{a_{m,n} : m, n \in \mathbb{N}\} \cup \{b_m : m \in \mathbb{N}\}$ topologized so that each point $a_{m,n}$ is isolated and a basic neighborhood of $b_m$ consists of $b_m$ itself and $a_{m,n}$ for all but finitely many $n$. Let $\sim$ be the equivalence relation on $X$ such that all $b_m$ are equivalent to each other and each $a_{m,n}$ is equivalent only to itself. Then $X/\sim$ is homeomorphic to the countable fan space.\(^8\)

(9) Consider the set $X = \{a_{m,n} : m, n \in \mathbb{N}\} \cup \{b_m : m \in \mathbb{N}\} \cup \{d_m : m \in \mathbb{N}\} \cup \{c\}$ topologized so that:
• points $a_{m,n}$ are isolated;
• a basic neighborhood of $b_m$ consists of $b_m$ itself and $a_{m,n}$ for all but finitely many $n$;
• a basic neighborhood of $c$ consists of $c$ itself and all but finitely many $d_n$.
Let $\sim$ be the equivalence relation on $X$ such that $b_m \sim d_m$ for every $m$, and other points are equivalent only to themselves. Then $X/\sim$ is homeomorphic to the Arens space.\(^9\)

7. Properties of mappings

A composition of two mappings with properties from Definition 1 is a mapping of the same type. A mapping inverse to a homeomorphism is a homeomorphism. A mapping inverse to a condensation is open (but not continuous unless when both the direct and inverse mapping are homeomorphisms).

If $Z \subset X$ and $f : X \to Y$ is continuous, or an homeomorphism, or an homeomorphic embedding, then so is the restricted mapping $f|Z : Z \to f(Z)$. Other properties from Definition 1 are not necessarily preserved by restrictions.

\(^8\)See “Countability conditions and convergent sequences”, Example 20
\(^9\)See “Countability conditions and convergent sequences”, Example 22
8. **Additional reading**

Sections 18 (about continuity) and 22 (about the quotient topology) of Munkres’s book. In Section 18, pay attention to Theorem 18.2(f): “the local formulation of continuity”, and Theorem 18.3: “The pasting lemma”. Try exercises 2,3,4,5.

In Section 22, there is a topic not discussed in this handout: quotient mappings and products (Example 7). Consider this topic optional (or return to it later, when we will study products of mappings). Try exercises 2,3,4. In particular, Exercise 2(b) is about a useful class of mappings called *retractions*. 